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On weighing matrices with square weights

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Abstract

We give a new construction for a known family of weighing matrices using the 2-adjugate method of Vartak and Patwardhan. We review the existence of $W(n, k^2)$, $k = 1, \dots, 12$, giving new results for $k = 8, \dots, 12$.

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On Weighing Matrices with Square Weights

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We give a new construction for a known family of weighing matrices using the 2-adjugate method of Vartak and Patwardhan. We review the existence of $W(n, k^2)$, $k = 1, \dots, 12$, giving new results for $k = 8, \dots, 12$.

1. Introduction

A weighing matrix, $W = W(n, k)$, is a square matrix of order n , with entries 0, +1, or -1, with the property that the scalar product of distinct rows is zero. Thus

$$WW^T = kI_n.$$

It has been conjectured that

Conjecture 1. (Seberry). *When $n \equiv 0 \pmod{4}$ there exists a $W(n, k)$ for all $k \leq n$.*

We discuss the conjecture

Conjecture 2. *For given k there exists an integer n_0 , so that for every $n > n_0$, there exists a $W(n, k)$.*

It is known that

THEOREM 1 (Geramita, Geramita and Seberry[1975/76]). *If a $W(n, k)$ exists and n is odd, then k is a square and*

$$(n-k)^2 - (n-k) + 2 > n.$$

We know that

(a) a $W(m^2 + m + 1, m^2)$ exists whenever m is a prime power and only if m is the order of a projective plane;

(b) given any integer k^2 there exists an n dependent on k^2 so that for every $m > n$ a $W(m, k^2)$ exists;

(c) there exists a

(i) $W(m, 4)$, where $m > 3$ (except $m=5, 9$, which do not exist);

(ii) $W(m, 9)$, where $m > 21$ ($m = 11$ does not exist and $m = 17, 21$ are undecided; $m = 15$ exists, Gibbons and Mathon[1986] found a $W(19, 9)$);

(iii) $W(m, 16)$, where $m > 35$ ($m = 17, 19$ do not exist and $m = 23, 25, 27, 29, 33, 35$ are undecided; $m = 31$ exists);

iv) $W(m, 25)$, where $m > 81$ ($m = 27, 29$ do not exist-- there are many undecided cases);

(v) $W(m, 36)$, where $m > 137$; ($m = 37, \dots, 41$ do not exist)

(vi) $W(m,49)$, where $m > 199$; ($m = 49, \dots, 58$ do not exist).

The proof of these results is in Geramita and Seberry [1979, p324-325].

We present a construction for weighing matrices using the 2-adjugation method of Vartak and Patwardhan and present results for $W(n,64), W(n,81), W(n,100), W(n,121), W(n,144)$. This allows us to say

(vii) $W(n,64)$ exist for $n > 135$ and $W(2n,64)$ exist for $2n > 31$ (Lemma 8); $W(n,64)$ for $n = 65, \dots, 71$ do not exist;

(viii) $W(n,81)$ for $n > 377$, $W(4n,81)$ for $n > 39$, $W(130,81)$ and $W(91,81)$ exist (Section 1 (a)); $W(n,81)$ for $n = 81, \dots, 89$ do not exist;

(ix) $W(n,100)$ exist for all $n > 336$ and $W(2n,100)$ exist for all $n > 59$. (Lemma 10); $W(n,100)$ for $n = 101, \dots, 109$ do not exist;

(x) $W(n,121)$ exist for all $n > 513$, $W(2n,121)$ exist for $2n > 381$. $W(4n,121)$ exist for $4n > 259$, $W(8n,121)$ exist for $8n > 127$. (Lemma 11); $W(n,121)$ do not exist for $n = 121, \dots, 131$;

(xi) $W(n,144)$ exists for all $n > 649$, $W(2n,144)$ exists for all $2n > 377$; (Lemma 12); $W(n,144)$ do not exist for $n = 145, \dots, 155$.

2. Known results

We use the following results

THEOREM 2. (Dandawate). *If $4t$ is the order of an Hadamard matrix then there exists a $W(2t(4t-1), 4t^2)$.*

THEOREM 3. (Geramita and Seberry) *Let r be any number of the form $2^a \cdot 10^b \cdot 26^c \cdot 5^d \cdot 13^e$, and let n be any integer $> 2^a \cdot 10^b \cdot 26^c \cdot 6^d \cdot 14^e$, a, b, c, d, e nonnegative integers. Then there exist (a) orthogonal designs of order $4n$ and types $(1,1,r,r)$ and $(1,4,r,r)$; (b) an orthogonal design of order $2n$ and type (r,r) or a $W(2n,2r)$.*

COROLLARY 4. *Let $2n$ be any integer > 119 then there exists a $W(2n,100)$.*

We use the fact that if there is an orthogonal design of type $(1,k)$ in order n there is an orthogonal design of type $(1,1,k,k)$ in order $2n$. Hence a $W(2n,2k+1)$ exists.

By the results of Geramita and Seberry[1979, Chapter 8] and Seberry[1982] we have

THEOREM 5. *Every orthogonal design of type $(1,k)$, $k < n-1$, exists in order $n = 2^t, 2^{t+1} \cdot 3, 2^{t+1} \cdot 5, 2^{t+1} \cdot 7, 2^{t+1} \cdot 9, 2^{t+1} \cdot 15, 2^{t+1} \cdot 21$, t a positive integer. Hence $W(2n,81)$ and $W(2n,121)$ exist for those m, n provided $n > 40, m > 60$. Thus $W(16a,81)$ exist for $a > 5$ and $W(16b,121)$ exist for all $b > 7$ (since by Theorem 1.149 of Geramita and Seberry an $OD(1,30)$ exists in every order $4n$, $n \geq 11$ we have $OD(1,1,2,60,60)$ and hence $W(16n,121)$ for $n \geq 11$).*

Also from Theorem 1.149 of Geramita and Seberry we have

LEMMA 6. *Orthogonal designs of type $(1,40)$ exist in all orders $4n$, $n > 10$. Thus $W(8n,81)$ exist $n > 10$.*

Since there are Golay sequences of length 40 we have

LEMMA 7. *There is an orthogonal design of type $(1,1,40,40)$ in all orders $4n$, $n \geq 40$. Thus $W(4n,81)$ exist for $n \geq 40$.*

3. Some $W(n, k^2)$

3.1. Case $W(n, 64)$

The following exist:

$W(2n, 64)$, $n > 31$ (Corollary 4),
 $W(120, 64)$ (Dandawate),
 $W(73, 64)$ (Section 1, (i)).

Thus we have

LEMMA 8. *There exist $W(n, 64)$ for $n > 135$.*

3.2. Case $W(n, 81)$

The following exist:

$W(4n, 81)$, $n > 39$ (Lemma 7)
 $W(91, 81)$, (complement of $PG(9, 2)$)
 $W(130, 81)$, (from $W(10, 9)$ and $W(13, 9)$)
 $W(169, 81)$, (from $W(13, 9)$).

Thus we have

LEMMA 9. *There exist $W(n, 81)$ for $n > 337$.*

3.3. Case $W(n, 100)$

$W(2n, 100)$ exist for all $n > 59$, (Corollary 4). $W(m, 25)$ and $W(n, 4)$ give a $W(mn, 100)$ so since $W(m, 25)$ exist for $m \in \{26, 28, 31, 32, 36, 40, 42, 44, 48, \text{ and all even orders } > 51\}$ and $W(n, 4)$ exist for $n \in \{4, 6, 7, 8, 10, 11, \dots\}$ we have $W(mn, 100)$ for $mn \in \{104, 112, 217\}$. Thus we have

LEMMA 10. *There exist $W(n, 100)$ for all $n > 336$. $W(2n, 100)$ exist for $2n = 104, 112$ and all $2n > 119$.*

3.4. Case $W(n, 121)$

$W(133, 121)$ exists ((i), section 1).
 $W(16t^2, 121)$ exist for all $t > 2$ since $W(4t, 11)$ exists.
 $W(16b, 121)$ exists for all $b > 7$ (from Theorem 5).
 $W(122, 121)$ exists since 121 is a prime power.

We construct a $W(132, 121)$. Let A be a $(1, 11)$ design in order 12, replace the first variable by J_{11} , the matrix of ones, and the second variable by B where $B = (b_{ij}) = (\chi(j-i))$, χ the quadratic character with $\chi(0)=0$. Then the new matrix is the required $W(132, 121)$.

Thus we have

LEMMA 11. *There exist $W(n, 121)$ for all $n > 513$. $W(2n, 121)$ exist for $2n > 381$. $W(4n, 121)$ exist for $4n$*

> 259. $W(16n, 121)$ exist for $16n > 127$.

3.5 Case $W(n, 144)$

Since there are 4-complementary sequences of length 9 and weight 36 and Golay sequences of length 4, we have 4-complementary sequences of length 36 and weight 144. Thus we have

$W(4n, 144)$ exist for all $n > 35$.

$W(273, 144)$ exists as it is $W(13, 9) \times W(21, 16)$.

$W(234, 144)$ exists as it is $W(18, 16) \times W(13, 9)$.

Hence

LEMMA 12. $W(n, 144)$ exists for all $n > 649$. $W(2n, 144)$ exists for all $2n > 377$.

4. A new construction for weighing matrices using 2-adjugates

Vartak and Patwardhan[1971] and Patwardhan and Vartak[1980] constructed the 2-adjugate mod 2 class of designs for unreduced BIBDs and symmetrical BIBDs with $\lambda = 1$. Patwardhan, Dandawate and Vartak[1984] applied the technique of 2-adjugation to a $(0, 1, -1)$ matrix and obtained a class of generalized PBIBDs with two associate classes and with triangular association scheme with unequal block sizes.

In this section we use the concept of 2-adjugates to obtain a construction for weighing matrices.

The 2-adjugate of a $(0, 1, -1)$ matrix N of order $(v \times b)$ is defined to be the matrix whose elements are the determinants of all possible 2×2 submatrices of N , arranged in lexicographic order. The 2-adjugate N^* of a matrix N whose entries come from a group is defined to be the matrix whose entries are formed by taking formal determinants of all possible 2×2 submatrices of N . Here by formal determinant we mean the following:

If the submatrix is

$$\begin{matrix} x, y \\ z, w \end{matrix}$$

then the corresponding entry of N^* is $xw - yz$.

Notice that in the special case when $N = GH(p, EA(p))$, the generalized Hadamard matrix of order p , p a prime power, the $(i, j)^{th}$ entry of N is $T^{(i-1)(j-1)}$ and hence by replacing the group elements by the corresponding matrix representation the 2-adjugate of any arbitrary two by two submatrix

$$\begin{matrix} T^{(i-1)(s-1)} & T^{(i-1)(t-1)} \\ T^{(j-1)(s-1)} & T^{(j-1)(t-1)} \end{matrix}$$

with $i < j$ is $T^{(i-1)(s-1)+(j-1)(t-1)} - T^{(i-1)(t-1)+(j-1)(s-1)}$.

An arbitrary row of H^* obtained from the i^{th} and j^{th} rows of H is given by

$$T^{(i-1)(t-1)+(j-1)(s-1)} - T^{(i-1)(s-1)+(j-1)(t-1)}; t=1, \dots, p-1; s=t+1, \dots, p.$$

We prove the following

LEMMA 13. Let H^* be the 2-adjugate formed from $H = GH(p, EA(p))$, the generalized Hadamard matrix

of size p , p a prime power, then the inner product of any distinct rows of H^* obtained from the i^{th} and j^{th} rows and k and l^{th} rows of H respectively is zero and the inner product of a row obtained from the i^{th} and j^{th} rows of H with itself is $p^2I - pJ$.

Proof:

The inner product of two rows of H^* is given by

$$\begin{aligned} & \sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} [T^{(i-1)(t-1)+(j-1)(s-1)} \cdot T^{(i-1)(s-1)+(j-1)(t-1)}] \\ & \cdot [T^{p-(k-1)(t-1)-(r-1)(s-1)} \cdot T^{p-(k-1)(s-1)-(r-1)(t-1)}] \\ & = \sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(t-1)+(j-r)(s-1)} + \sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(s-1)+(j-r)(t-1)} \\ & - \sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-r)(s-1)+(j-k)(t-1)} - \sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-r)(t-1)+(j-k)(s-1)}. \end{aligned}$$

Notice that the first and second terms give

$$\sum_{t=1}^{i=(p-1)} \sum_{s=(t+1)}^{s=p} T^{(i-k)(t-1)} \cdot T^{(j-r)(s-1)} + \sum_{s=1}^{s=(p-1)} \sum_{t=(s+1)}^{t=p} T^{(i-k)(t-1)} \cdot T^{(j-r)(s-1)}$$

Which gives us

$$\sum_{t=1}^{i=p} [T^{(i-k)(t-1)} \cdot \sum_{s=1}^{s=p} T^{(j-r)(s-1)}] (s \neq t).$$

Now we add and subtract

$$\sum_{t=1}^{i=p} T^{(i-k)(t-1)+(j-r)(t-1)}.$$

Similarly we evaluate third and fourth terms and get:

$$\sum_{t=1}^{i=p} T^{(i-k)(t-1)} \cdot \sum_{s=1}^{s=p} T^{(j-r)(s-1)} - \sum_{t=1}^{i=p} T^{(t-1)(i+j-k-r)} - \sum_{t=1}^{i=p} T^{(i-r)(s-1)} \cdot \sum_{s=1}^{s=p} + \sum_{t=1}^{i=p} T^{(t-1)(i+j-k-r)}.$$

Which is $p^2I - pJ$ when $i=k$ and $j=r$, and zero otherwise.

THEOREM 14. Weighing matrices $W(p^2(p-1), p^2)$ exist for all prime powers p .

Proof: Form H^* from $H = GH(p, EA(p))$ the generalized Hadamard matrix. Obtain G from H^* by replacing the entries of H^* by the corresponding $(0, 1, -1)$ matrices. Form a matrix $S = I \times J_p$. Then

$$\begin{bmatrix} G & S \\ S & G \end{bmatrix}$$

is the required weighing matrix.

EXAMPLE. Let $H = GH(3, EA(3))$ given by

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$$

Replace ω^i by T^i and 1 by the identity matrix of order 3, where T is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Now form H*:

$$\begin{bmatrix} 1-\omega & 1-\omega^2 & \omega^2-\omega \\ 1-\omega^2 & 1-\omega & \omega-\omega^2 \\ \omega^2-\omega & \omega-\omega^2 & \omega^2-\omega \end{bmatrix}$$

which gives G:

$$\begin{array}{ccccccc} - & 1 & 0 & - & 0 & 1 & 0 & - & 1 \\ 0 & - & 1 & 1 & - & 0 & 1 & 0 & - \\ 1 & 0 & - & 0 & 1 & - & - & 1 & 0 \\ \\ - & 0 & 1 & - & 1 & 0 & 0 & 1 & - \\ 1 & - & 0 & 0 & - & 1 & - & 0 & 1 \\ 0 & 1 & - & 1 & 0 & - & 1 & - & 0 \\ \\ 0 & - & 1 & 0 & 1 & - & 0 & - & 1 \\ 1 & 0 & - & - & 0 & 1 & 1 & 0 & - \\ - & 1 & 0 & 1 & - & 0 & - & 1 & 0 \end{array}$$

Now form S:

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array}$$

Now form W:

$$\begin{bmatrix} G & S \\ S & G \end{bmatrix}$$

Note that in this particular case we have $W(18, 3^2)$. We also note that in the case of order 3 we have using

$$GW(4,3,2) = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & - \\ 1 & - & 0 & 1 \\ 1 & 1 & - & 0 \end{bmatrix}$$

that

$$\begin{array}{cccccccccccc} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & - & - & - & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & - & - & - \\ 0 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{c} 1 \ 0 \ 1 \ - \\ 1 \ 0 \ 1 \ - \\ 1 \ 0 \ 1 \ - \end{array}$$

$$\begin{array}{c} 1 \ - \ 0 \ 1 \\ 1 \ - \ 0 \ 1 \\ 1 \ - \ 0 \ 1 \end{array} \quad G$$

$$\begin{array}{c} 1 \ 1 \ - \ 0 \\ 1 \ 1 \ - \ 0 \\ 1 \ 1 \ - \ 0 \end{array}$$

is a $W(13,9)$.

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